ON THE STRESSES IN A PIECEWISE HOMOGENEOUS MEDIUM

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A method is given for solution of the problem of stresses in an infinite, piecewise homogeneous medium, weakened by a circular opening. Analytical continuation of Kolosov-Muskhelishvili potentials is used to reduce the problem to the form, allowing a direct solution by means of power series. An infinite system of linear algebraic equations composed of the coefficients of the expansion is constructed and solved by numerical methods for the case of uniaxial tensile deformation of a plate, with the elastic parameters of the material known.

Let us imagine an infinite continuous plate made of two materials possessing distinct elastic properties, sharing a common rectilinear boundary. Let one of the parts with elastic parameters λ_1, μ_1 say, occupy the lower half of the plane $z = x + iy$, and the other part with parameters λ_2, μ_2, – the upper half. Let us also assume that the piecewise homogeneous medium is weakened by a circular opening, and is subject to external stresses applied at the boundary of the opening and at infinity. We shall assume the radius of the opening to be a unit radius with its center at the origin of the coordinates. Then, the real axis x less the $(-1, +1)$ segment will become the boundary which we shall denote by L. Further, we shall denote the lower and the upper half-plane without the respective semi-circles by S^- and S^+, respectively, and the lower and the upper semicircular boundary by γ_1 and γ_2.

We shall try to determine complex potentials φ and ψ holomorphic in S^- and S^+, respectively, and identical indices will be used for the potentials and the corresponding elastic constants. We have the following boundary value problem:

$$\varphi_1(t) + (t - \bar{t})\,\overline{\varphi_1'(t)} + \overline{\chi_1(t)} = f(t) \qquad \text{on } \gamma_1$$

$$\varphi_2(t) + (t - \bar{t})\,\overline{\varphi_2'(t)} + \overline{\chi_2(t)} = f(t) \qquad \text{on } \gamma_2 \tag{1}$$

$$\left.\begin{aligned} &\varphi_1(t) + \overline{\chi_1(t)} = \varphi_2(t) + \overline{\chi_2(t)} \\ &\lambda\,[\varkappa_1, \varphi_1(t) - \overline{\chi_1(t)}] = \varkappa_2\varphi_2(t) - \overline{\chi_2(t)} \end{aligned}\right\} \qquad \text{on } L \tag{2}$$

Here, $f(t)$ is some specified function of a point on the unit circle

$$\chi(z) = z\varphi'(z) + \psi(z), \qquad \lambda = \mu_2/\mu_1 \tag{3}$$

Adding and subtracting Equations (2) and changing over to conjugate values in the second of them. we obtain

$$\left.\begin{aligned} &(1 + \lambda\varkappa_1)\,\varphi_1(t) + (1 - \lambda)\,\overline{\chi_1(t)} = (1 + \varkappa_2)\,\varphi_2(t) \\ &(\varkappa_2 + \lambda)\,\chi_1(t) + (\varkappa_2 - \lambda\varkappa_1)\,\overline{\varphi_1(t)} = (1 + \varkappa_2)\,\chi_2(t) \end{aligned}\right\} \qquad \text{on } L \tag{4}$$

It can easily be seen that the functions φ_1, γ_1, which can be defined in S^+ by

$$\left.\begin{aligned}(1+\lambda\varkappa_1)\varphi_1(z) &= (\lambda-1)\bar{\chi}_1(z)+(1+\varkappa_2)\varphi_2(z)\\ (\varkappa_2+\lambda)\chi_1(z) &= (\lambda\varkappa_1-\varkappa_2)\bar{\varphi}_1(z)+(1+\varkappa_2)\chi_2(z)\end{aligned}\right\} \quad \text{for } z \text{ in } S^+ \qquad (5)$$

extend, by analytic continuation, the values of the complex potentials φ_1, χ_1, into S^+, across the line L. In other words, functions φ_1, χ_1, extended over S^+ by Equations (5) will, by (4), be holomorphic over the whole plane with a circular opening.

This extended domain we shall now call S, and the functions φ_1, χ_1 holomorphic on it, φ and χ.

Now we can express φ_2, γ_2 in terms of just introduced φ and γ.

Use of the new notation will transform (5) into

$$\begin{aligned}(1+\varkappa_2)\varphi_2(z) &= (1+\lambda\varkappa_1)\varphi(z)+(1-\lambda)\bar{\chi}(z)\\ (1+\varkappa_2)\chi_2(z) &= (\varkappa_2+\lambda)\chi(z)+(\varkappa_2-\lambda\varkappa_1)\bar{\varphi}(z)\end{aligned} \qquad (6)$$

when z is on S^+.

If we now substitute φ_2, χ_2 obtained from the previous equations into the second condition of (1), then the following boundary conditions:

$$\varphi(t)+(t-\bar{t})\overline{\varphi'(t)}+\overline{\chi(t)} = f(t) \quad \text{on } \gamma_1 \qquad (7)$$

$$\varphi(t)+(t-\bar{t})\overline{\varphi'(t)}+\overline{\chi(t)}+\alpha\varphi(\bar{t})+\beta\overline{\chi(t)}+\gamma\bar{\chi}(t)+\gamma(t-\bar{t})\chi'(\bar{t}) = (1+\alpha)f(t) \quad \text{on } \gamma_2$$

where

$$\alpha = \frac{\mu_1\varkappa_2-\mu_2\varkappa_1}{\mu_1+\mu_2\varkappa_1}, \qquad \gamma = \frac{\mu_1-\mu_2}{\mu_1+\mu_2\varkappa_1}, \qquad \beta = \alpha-\gamma \qquad (8)$$

will yield the functions φ and γ holomorphic on S.

We shall proceed to solve (7) by assuming that on S,

$$\varphi(z) = \sum_{k=1}^{\infty} a_k z^{-k}, \qquad \chi(z) = \sum_{k=0}^{\infty} c_k z^{-k} \qquad (9)$$

Now, assuming that the series obtained from (9) by differentiation converge uniformly on the circle, let us collect the left-hand sides of Equations (7). The result of this will be

$$\varphi(t)+(t-\bar{t})\overline{\varphi'(t)}+\overline{\chi(t)} = \bar{c}_0+\sum_{k=1}^{\infty} a_k t^{-k}+\sum_{k=1}^{\infty}\Omega_k' t^k$$

$$\alpha\varphi(\bar{t})+\beta\overline{\chi(t)}+\gamma\bar{\chi}(t)+\gamma(t-\bar{t})\chi'(\bar{t}) = \alpha\bar{c}_0+\sum_{k=1}^{\infty}\bar{c}_k t^{-k}+\sum_{k=1}^{\infty}\Omega_k'' t^k$$

$$(t = e^{i\vartheta}, \quad 0 \leqslant \vartheta \leqslant 2\pi) \qquad (10)$$

where

$$\left.\begin{aligned}\Omega_k' &= \bar{c}_k+k\bar{a}_k-(k-2)\bar{a}_{k-2}\\ \Omega_k'' &= \alpha a_k+\beta\bar{c}_k+\gamma[kc_k-(k-2)c_{k-2}]\end{aligned}\right. \qquad (k = 1, 2, 3, \ldots) \qquad (11)$$

In (11), the term containing the factor $(k - 2)$ should be omitted, when the value of $k = 1$. In the argument following, we shall make use of the equalities

$$\frac{1}{\pi i}\int_0^\pi\left\{\sum_{k=1}^{\infty}\lambda_k t^k\right\}t^{-n-1}\,dt = \lambda_n+\frac{1}{\pi i}\sideset{}{'}\sum_{k=1}^{\infty}\frac{(-1)^{k-n}-1}{k-n}\lambda_k \qquad (n \geqslant 0)$$

$$\frac{1}{\pi i}\int_0^\pi\left\{\sum_{k=1}^{\infty}\lambda_k t^{-k}\right\}t^{-n-1}\,dt = \lambda_{-n}-\frac{1}{\pi i}\sideset{}{'}\sum_{k=1}^{\infty}\frac{(-1)^{k+n}-1}{k+n}\lambda_k \qquad (n \leqslant 0) \qquad (12)$$

$$\frac{1}{\pi i}\int_0^\pi t^{-n-1}\,dt = \left\{\ [1-(-1)^n]/\pi i n \qquad (n \neq 0)\right.$$

which are easily obtained by integrating by parts the respective sums.

Also, we shall introduce the function $g(t)$ defined on the unit circle as follows:

$$g(t) = f(t) \quad \text{on } \gamma_1, \qquad g(t) = (1+\alpha) f(t) \quad \text{on } \gamma_2 \tag{13}$$

If now

$$g(t) = \sum_{k=-\infty}^{\infty} A_k t^k, \qquad A_k = \frac{1}{2\pi i}\int_0^{2\pi} g(t)\, t^{-k-1}\, dt \qquad (k = 0, \pm 1, \pm 2, \dots) \tag{14}$$

then substituting (14) into (7) and comparing the coefficients of like powers of $t^n (n = 0, \pm 1, \pm 2, \dots)$, we shall, using (12), obtain

$$(1 + {}^1\!/_2\, \alpha)\bar{c}_0 - \frac{\gamma}{2\pi i}\sum_{k=1}^{\infty}\frac{(-1)^k - 1}{k}\bar{c}_k + \frac{1}{2\pi i}\sum_{k=1}^{\infty}\frac{(-1)^k - 1}{k}\Omega_k'' = A_0$$

$$\Omega_n' + {}^1\!/_2\,\Omega_n'' + \frac{1}{2\pi i}\sum_{k=1}^{\infty}{}'\frac{(-1)^{k-n} - 1}{k-n}\Omega_k'' - \frac{\gamma}{2\pi i}\sum_{k=1}^{\infty}\frac{(-1)^{k+n} - 1}{k+n}\bar{c}_k -$$

$$- \frac{\alpha}{2\pi i}\frac{(-1)^n - 1}{n}\bar{c}_0 = A_n$$

$$\alpha_n + {}^1\!/_2\,\gamma\bar{c}_n - \frac{\gamma}{2\pi i}\sum_{k=1}^{\infty}{}'\frac{(-1)^{k-n} - 1}{k-n}\bar{c}_k + \frac{1}{2\pi i}\sum_{k=1}^{\infty}\frac{(-1)^{k+n} - 1}{k+n}\Omega_k'' +$$

$$+ \frac{\alpha}{2\pi i}\frac{(-1)^n - 1}{n}\bar{c}_0 = A_{-n} \tag{15}$$

$$(n = 1, 2, 3, \dots)$$

The set of Equations (15) represents an infinite system of linear algebraic equations in terms of the unknown coefficients of the series (9). For particular values of elastic parameters of the materials, e.g. $\alpha = 0$, when the elastic constants are connected by the relation $\mu_1\varkappa_2 = \mu_2\varkappa_1$, we can using elementary operations, eliminate from (15) all a_k $(k = 0, 1, 2, \dots)$ and obtain a set of equations containing only the unknowns c_k $(k = 0, 1, 2, \dots)$. For the arbitrary elastic parameters however, the attempt at reducing the system would not be expedient.

For numerical methods to be applied, a finite system is required and this can be obtained from (15) for some $n = N$ in the following manner: we shall define both unknowns c_n and a_n in terms of one variable x

$$x_{2n} = c_n \quad (n = 0, 1, \dots), \qquad x_{2n-1} = a_n \quad (n = 1, 2, \dots) \tag{16}$$

and shall write the reduced system as follows:

$$(1 + \alpha/2)\bar{x}_0 - \gamma\sum_{k=1}^{N}\delta_k\bar{x}_{2k} + \sum_{k=1}^{N}\delta_k\Omega_{2k-1} = A_0$$

$$\Omega_{2n} + \frac{1}{2}\Omega_{2n-1} + \sum_{k=1}^{N}{}'\delta_{k-n}\Omega_{2k-1} - \gamma\sum_{k=1}^{N}\delta_{k+n}\bar{x}_{2n} - \alpha\delta_n\bar{x}_0 = A_n$$

$${}_{\imath-1} + \frac{\gamma}{2}\bar{x}_{2n} - \gamma\sum_{k=1}^{N}{}'\delta_{k-n}\bar{x}_{2k} + \sum_{k=1}^{N}\delta_{k+n}\Omega_{2k+1} + \alpha\delta_n\bar{x}_0 = A_{-n} \tag{17}$$

$$(n = 1, \dots, N)$$

where

$$\delta_\nu = \frac{(-1)^\nu - 1}{2\pi i\nu} \qquad (\nu = 1, 2, \dots)$$

$$\Omega_{2k} = \bar{x}_{2k} + k\bar{x}_{2k-1} - (k-2)\bar{x}_{2k-5} \qquad (k = 1, 2, \dots)$$

$$\Omega_{2k-1} = \alpha x_{2k-1} + \beta\bar{x}_{2k} + \gamma[k x_{2k} - (k-2) x_{2k-4}]$$

Using the above formulas to solve (15) we can determine both pairs of the Kolosov-Muskhelishvili potentials and hence, find the remaining unknowns. As a particular case of some practical value, we shall quote the sum of normal stresses on the contour of the opening, (assuming the absence of stresses at infinity), the formulas for which are

on $\gamma_1 \, (\pi \leqslant \vartheta \leqslant 2\pi)$

$$\sigma_r + \sigma_\vartheta = -4 \sum_{k=1}^{\infty} k \, [x_{2k-1}' \cos (k+1) \vartheta + x_{2k-1}'' \sin (k+1) \vartheta]$$

on $\gamma_2 \, (0 \leqslant \vartheta \leqslant \pi)$

$$\sigma_r + \sigma_\vartheta = -4 \sum_{k=1}^{\infty} k \left[\left(\frac{x_{2k-1}'}{1+\alpha} + \delta x_{2k}' \right) \cos (k+1) \vartheta + \left(\frac{x_{2k-1}''}{1+\alpha} - \delta x_{2k}'' \right) \sin (k+1) \vartheta \right] \quad (18)$$

$$x_k = x_k' + i x_k'' \qquad (k = 0, 1, \ldots), \qquad \delta = \frac{\mu_1 - \mu_2}{\mu_1 (1 + \varkappa_2)}$$

We shall not consider here the theoretical aspects of the problem. To illustrate the method, we shall use the case of a nonhomogeneous plate with an opening, subject to a uniform tensile stress applied at infinity in the direction of the x-axis.

By denoting the tensile stress by p we obtain

$$f(t) = {}^1\!/_2 \, p \, (t^{-1} - t) \qquad (t = e^{i\vartheta}, \; 0 \leqslant \vartheta \leqslant 2\pi) \qquad (19)$$

Furier coefficients of the function $g(t)$ determinable by (13), are

$$A_1 = -\left(1 + \frac{\alpha}{2}\right) \frac{p}{2}, \qquad A_{-1} = \left(1 + \frac{\alpha}{2}\right) \frac{p}{2}, \qquad A_n = \frac{\alpha}{2\pi i} \frac{(-1)^{n-1} - 1}{n^2 - 1} p \qquad (20)$$
$$(n = 0, \pm 2, \pm 3, \ldots)$$

We shall assume the Poisson coefficients to be identical for the purpose of our calculations. Hence $\varkappa_1 = \varkappa_2 = \varkappa$. In this case, the constants α, β, .. shown above can be expressed in terms of $\varkappa$ and the elastic moduli λ by

$$\alpha = \frac{\varkappa (1 - \lambda)}{1 + \lambda \varkappa}, \quad \beta = \frac{(\varkappa - 1)(1 - \lambda)}{1 + \lambda \varkappa}, \quad \gamma = \frac{1 - \lambda}{1 + \lambda \varkappa}, \quad \delta = \frac{1 - \lambda}{1 + \varkappa} \qquad \left(\lambda = \frac{E_2}{E_1}\right) \quad (21)$$

The reduced system (17) was programed and solved on the БЭСМ - 2 (**BESM - 2**) computer by D.P. Vakhtangadze for various values of the parameter λ at $\varkappa = 2$ (Poisson coefficient for both materials was assumed to be 0.25).

The calculations have shown that the boundary conditions of the problem (7) were satisfied to the acceptable degree of accuracy already for $N = 8$. This is equivalent to solving a set of 34 real equations. Below, the right-hand side shows the values of the coefficients k_1, k_2 of the stress concentration

$$k = p^{-1} \max (\sigma_r + \sigma_\vartheta)$$

on the lower and upper semi-circle

$\lambda =$	0.25	0.50	0.75
$k_1 =$	3.5925	3.2854	3.1041
$k_2 =$	2.5885	2.7605	2.8952

It should be noted that for the homogeneous case ($\lambda = 1$), we have $k_1 = k_2 = 3$.

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